The stiffness of plates

1. Introduction

The word ‘plate’ is a collective term for elements in which forces can be transferred in two directions. Floors, walls, bridge slabs and laminates are all plates.

Plates can be loaded in their plane and perpendicular to their plane. The membrane behaviour describes how a plate reacts to loading in its plane. The bending behaviour\(^1\) describes how a plate reacts to loading perpendicular to its plane.

To describe the behaviour we will need the rigidity of the plate. Rigidity is a measure for the resistance of an element against deformation (thus strain \(\varepsilon\) and curvatures \(\kappa\)).

Once the rigidity is known, Diamonds will use it in the Finite Element Method\(^2\). The FEM chops the structure into a finite number of elements, which will then be logically linked to each other, allowing Diamonds to calculate the displacements of the structure. The stresses (and resulting sectional forces) can then be derived from the displacements because of these relations between them:

- The kinematic equations give the relations between the displacements and the deformations.
- The constitutive equations give information on the material behaviour, by providing the relations between the stresses and the strains.
- The equilibrium equations give the relations between the loads and the stresses.

With the aim of determining the rigidity of plates, the focus of this document lies on the constitutive equation. More information on the other relations can be found in for example [2].

\(^{1}\) The shear behaviour is not handled in this document since only Thin Plate Theory (neglecting the shear deformation of the plate) is implemented in Diamonds. The Thick Plate Theory (taking the shear deformation for plates into account) is not implemented. Thin Plate Theory is also called ‘Kirchoff Theory’, Thick Plate Theory is also called ‘Mindlin Theory’.

\(^{2}\) short ‘FEM’, synonym ‘displacement method’
2. An isotropic plate

The simplest form of a plate is an isotropic plate. The word ‘isotropic’ refers to the material behaviour and means ‘homogeneous in all directions’, like solid steel. §2.1 describes the stress-strain relation for an isotropic material so the sectional forces (membrane forces and bending moments) can be calculated in §2.2. The example in §2.3 shows how the formula are used.

2.1. Constitutive equations

2.1.1. Hooke’s law in 3D

If the material is subjected to a state of triaxial stress, associated normal strains will be developed in the material. The total strain in a direction equals the sum of all strains in that direction due to the stresses in each direction [4]:

\[
\begin{align*}
\varepsilon_x &= \sigma_{x,xx} + \sigma_{x,yy} + \sigma_{x,zz} \\
\varepsilon_y &= \sigma_{y,xx} + \sigma_{y,yy} + \sigma_{y,zz} \\
\varepsilon_z &= \sigma_{z,xx} + \sigma_{z,yy} + \sigma_{z,zz}
\end{align*}
\]

(1)

Taking Hooke’s law (\(\sigma = E \cdot \varepsilon\)) and Poisson’s ratio \(\nu\) into account, the expressions (1) transform into:

\[
\begin{align*}
\varepsilon_x &= \sigma_{x,xx} + \sigma_{x,yy} + \sigma_{x,zz} \\
\varepsilon_x &= \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \\
\varepsilon_x &= \frac{1}{E} \left( \sigma_x - \nu \sigma_y - \nu \sigma_z \right) \\
\varepsilon_y &= \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E} \\
\varepsilon_z &= \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \\
\varepsilon_z &= \frac{1}{E} \left( \sigma_x - \nu \sigma_y - \nu \sigma_y \right)
\end{align*}
\]

(2)
If we apply a shear stress $\tau_{xy}$ to the element, the material will deform only due to a shear strain $\gamma_{xy}$, that is $\tau_{xy}$ will not cause other strains in the material. Likewise $\tau_{yz}$ and $\tau_{xz}$ will only cause shear strains $\gamma_{yz}$ and $\gamma_{xz}$.

Hooke’s law relating shear stress and shear strain is:

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \quad \gamma_{xz} = \frac{1}{G}\tau_{xz} \tag{5}$$

Write equations (2), (3), (4) and (5) in matrix form (stress-strain) and Hooke’s law for a linear elastic material in 3D is obtained:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{E\cdot\nu}{E(1-\nu)} & \frac{E\cdot\nu}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{E\cdot\nu}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{E\cdot\nu} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{E\cdot\nu}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{E\cdot\nu} & 0 & 0 & 0 \\ \frac{1}{(1-2\nu)(1+\nu)} & \frac{1}{(1-2\nu)(1+\nu)} & \frac{1}{(1-2\nu)(1+\nu)} & \frac{E}{G} & 0 & 0 \\ \frac{1}{(1-2\nu)(1+\nu)} & \frac{1}{(1-2\nu)(1+\nu)} & \frac{1}{(1-2\nu)(1+\nu)} & 0 & \frac{E}{G} & 0 \\ \frac{1}{(1-2\nu)(1+\nu)} & \frac{1}{(1-2\nu)(1+\nu)} & \frac{1}{(1-2\nu)(1+\nu)} & 0 & 0 & \frac{E}{G} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{6}$$

2.1.2. Hooke’s law in 2D

If the plate is thin and there are no out-of-plane loads, it can be considered to be under plane stress. Then $\sigma_y = 0$, $\tau_{yz} = \tau_{xy} = 0$. Equation (6) simplifies to:

$$\begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & 1 & 0 & 0 & \varepsilon_x \\ 1 & \frac{1}{1-\nu^2} & 1 & 0 & \varepsilon_y \\ 0 & 0 & 1-\nu^2 & 2 & \gamma_{xz} \end{bmatrix} \begin{bmatrix} Q_{11} \\ Q_{12} \\ Q_{16} \\ Q_{16} \\ Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xz} \end{bmatrix} \tag{7}$$

The 3x3 matrix $[Q]$ in equation (7) is called the elasticity matrix.

Note: an alternative way to transform Hooke’s law from 3D to 2D is plane strain. But since Diamonds assumes plane stress, this is not treated.
2.2. Sectional forces

The resultant membrane forces and moments can be calculated as the sum of the stresses (equ. (7)) over the thickness $t$ of the plate:

$$
\begin{pmatrix}
N_{xx} \\
N_{zz} \\
N_{xz}
\end{pmatrix} = \int_{-t/2}^{t/2} \begin{pmatrix}
\sigma_{xx} \\
\sigma_{zz} \\
\tau_{xz}
\end{pmatrix} dy
$$

$$
\begin{pmatrix}
M_{xx} \\
M_{zz} \\
M_{xz}
\end{pmatrix} = \int_{-t/2}^{t/2} \begin{pmatrix}
\sigma_{xx} \\
\sigma_{zz} \\
\tau_{xz}
\end{pmatrix} y dy
$$

This results in the following matrix equations (with nor $\sigma_{xx}$ nor $\varepsilon$ depended of the thickness):

$$
\begin{pmatrix}
N_{xx} \\
N_{zz} \\
N_{xz}
\end{pmatrix} = t \frac{E}{1 - \nu^2} \begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1 - \nu
\end{pmatrix} \begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{zz} \\
\gamma_{xz}
\end{pmatrix} = t \cdot [Q] \begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{zz} \\
\gamma_{xz}
\end{pmatrix}
$$

(8)

$$
\begin{pmatrix}
M_{xx} \\
M_{zz} \\
M_{xz}
\end{pmatrix} = \frac{t^3}{12 (1 - \nu^2)} \begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1 - \nu
\end{pmatrix} \begin{pmatrix}
\kappa_{xx} \\
\kappa_{zz} \\
\rho_{xz}
\end{pmatrix} = \frac{t^3}{12} \cdot [Q] \begin{pmatrix}
\kappa_{xx} \\
\kappa_{zz} \\
\rho_{xz}
\end{pmatrix}
$$

(9)

Equation (8) expresses the membrane behaviour of an isotropic plate, equation (9) the bending behaviour.

Notes:

- The 3x3 matrix $[d]$ in equation (8) is called the plate membrane stiffness matrix. The 3x3 matrix $[D]$ in equation (9) is called the plate bending stiffness matrix.
- In Diamonds equations (8) and (9) are combined in one matrix equation. And $d_{16} = d_{26} = D_{16} = D_{26} = 0$ (more info in §3.2.2):

$$
\begin{pmatrix}
N_{xx} \\
N_{zz} \\
N_{xz}
\end{pmatrix} = \begin{pmatrix}
d_{11} & d_{12} & 0 \\
d_{12} & d_{22} & 0 \\
0 & 0 & d_{66}
\end{pmatrix} \begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{zz} \\
\gamma_{xz}
\end{pmatrix}
$$

(10)
2.3. Example: isotropic geometry – isotropic material

Concrete C25/30:
\[ E = 31476 \text{N/mm}^2, v = 0.2 \]

2.3.1. Manual calculation

We use equation (8) to describe the membrane behaviour.
\[
\begin{align*}
  d_{11} &= d_{22} = \frac{E \cdot t}{1-v^2} = \frac{31476 \text{N/mm}^2 \cdot 200 \text{mm}}{1 - 0.2^2} = 6557 \cdot 10^3 \text{kN/m} \\
  d_{12} &= d_{21} = \frac{E \cdot t}{1-v^2} = \frac{6557 \cdot 10^3 \text{kN/m}}{1 - 0.2^2} = 1311 \cdot 10^3 \text{kN/m} \\
  d_{66} &= \frac{1}{2} (1-v) \frac{E \cdot t}{1-v^2} = \frac{1}{2} (1-0.2) \cdot 6557 \cdot 10^3 \text{kN/m} = 2623 \cdot 10^3 \text{kN/m}
\end{align*}
\]

We use equation (9) to describe the bending behaviour.
\[
\begin{align*}
  D_{11} &= D_{22} = \frac{E \cdot t^3}{12(1-v^2)} = \frac{31476 \text{N/mm}^2 \cdot (200 \text{mm})^3}{12(1 - 0.2^2)} = 21858 \text{kNm} \\
  D_{12} &= D_{21} = \frac{E \cdot t^3}{12(1-v^2)} = \frac{6557 \cdot 10^3 \text{kN/m}}{1 - 0.2^2} = 4372 \text{kNm} \\
  D_{66} &= \frac{1}{2} (1-v) \frac{E \cdot t^3}{12(1-v^2)} = \frac{1}{2} (1-0.2) \cdot 21858 \text{kNm} = 8743 \text{kNm}
\end{align*}
\]

2.3.2. In Diamonds

The stiffness matrix of an isotropic plate in Diamonds gives the same results as calculated by hand:

\[
\begin{bmatrix}
  6557 & 1311 & 0 \\
  1311 & 5557 & 0 \\
  0 & 0 & 2623
\end{bmatrix}
\]

\[
\begin{bmatrix}
  21858 & 4372 & 0 \\
  4372 & 20308 & 0 \\
  0 & 0 & 8743
\end{bmatrix}
\]

Note: If you want to compare the stiffness matrix in Diamonds to manual calculations, make sure the correct standard (here EN 1992-1-1 [--]) is selected. Some materials have a different Young’s modulus depending on the standard/ national annex.
3. Other plate types

Many plate types cannot be handled as an isotropic plate. Stiffeners may occur and they can be different in two orthogonal directions (§3.1). Not all materials have isotropic properties, a material can have different properties in two mutually perpendicular directions. This type of material is called orthotropic (§3.2). A plate can also be composed of multiple orthotropic layers. This type of plate is called a laminate (§3.3).

3.1. Orthotropic geometry (single I-slab) – isotropic material

In this example we will replace the real shape of the plate with a solid plate taking the geometry of the stiffeners into account. This method is referred to as shape-orthotropy. It can be used for plates with repeating stiffeners with regular spacing.

The formula for the stiffness components are [2]:

\[
[d] = \begin{bmatrix}
d_{11} & d_{12} & d_{16} \\
d_{12} & d_{22} & d_{26} \\
d_{16} & d_{26} & d_{66}
\end{bmatrix} = \begin{bmatrix}
\frac{E \cdot t}{1 - v^2} & \frac{E \cdot A_a}{a} & v \cdot d_{22} \\
\frac{E \cdot A_a}{a} & \frac{E \cdot t}{1 - v^2} & 0 \\
v \cdot d_{22} & 0 & \frac{1}{2} (1 - v) \cdot d_{22}
\end{bmatrix}
\]

\[
[D] = \begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66,av}
\end{bmatrix} = \begin{bmatrix}
\frac{E \cdot I_{xx}}{a} & v \cdot D_{22} & 0 \\
v \cdot D_{22} & \frac{E \cdot t^3}{12(1 - v^2)} & 0 \\
0 & 0 & 0.5 \cdot G \cdot \frac{i_{12} + i_{21}}{2}
\end{bmatrix}
\]

The torsional rigidity $D_{66}$ needs special attention. Using here the formula for an isotropic plate will underestimate the torsional rigidity, because the stiffeners are neglected. The alternative is to calculate the torsional rigidities per units length in each direction ($i_{xx}$ and $i_{zz}$) and take the average, resulting in $D_{66,av}$. Formula for the rigidity per units length are given in Figure 4.
3.1.1. Manual calculation

\[ d_{11} = \frac{31476N/mm^2 \cdot 100mm}{(1 - 0.2^2)} + \frac{31476N/mm^2 \cdot 50000mm^2}{600mm} = 5902 \cdot 10^3 kN/m \]
\[ d_{22} = \frac{31476N/mm^2 \cdot 100mm}{(1 - 0.2^2)} = 3279 \cdot 10^3 kN/m \]
\[ d_{12} = d_{21} = 656 \cdot 10^3 kN/m \]
\[ d_{66} = \frac{1}{2} (1 - 0.2) \cdot 3279 \cdot 10^3 kN/m = 1312 \cdot 10^3 kN/m \]

\[ D_{11} = \frac{31476N/mm^2 \cdot 232500000 mm^4}{600mm} = 121970 kNm \]
\[ D_{22} = \frac{31476N/mm^2 \cdot (100mm)^3}{12(1 - 0.2^2)} = 2732 kNm \]
\[ D_{12} = D_{21} = 0.2 \cdot D_{22} = 0.2 \cdot 2732kNm = 546 kNm \]

To calculate the torsional rigidity in the z-direction \( i_{12} \), the I-section can be seen as a sum of rectangles [7]. In the \( x' \)-direction \( i_{21} \) we use the formula for an isotropic plate.

\[ i_{12} = \frac{i(e, e_1) + i(e_2, e_3 - e_4) + i(e_4, e_5)}{e_1} = \frac{i(100mm, 600mm) + i(100mm, 200mm) + i(100mm, 200mm)}{600mm} \]
\[ = 771470mm^3 \]
\[ i_{21} = \frac{e^3}{6} = \frac{(100mm)^3}{6} = 166667mm^3 \]
\[ D_{66,av} = 0.5 \cdot \frac{31476N/mm^2 \cdot (771470 + 166667)mm^3}{2(1 + 0.2)} = 3076kNm \]

3.1.2. In Diamonds

These values should be entered manually in Diamonds.
Note: Since the real shape of the plate is not known by Diamonds, no stresses nor reinforcement can be calculated.

3.2. Isotropic geometry – orthotropic material

Norway Spruce (Dinwoodie, 2000):

- $E_1 = 10\,700\,N/mm^2$ (x’-direction)
- $E_2 = 430N/mm^2$ (x’-direction)
- $G = 620N/mm^2$
- $\nu_{12} = 0.51, \nu_{21} = 0.02$

The formula for the stiffness components are [5]:

$$
[D] = \begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66,av}
\end{bmatrix}
= \begin{bmatrix}
\frac{E_1 \cdot t}{1 - \nu_{12} \nu_{21}} & \frac{E_2 \cdot t}{1 - \nu_{12} \nu_{21}} & 0 \\
\frac{E_1 \cdot t^3}{12(1 - \nu_{12} \nu_{21})} & \frac{E_2 \cdot t^3}{12(1 - \nu_{12} \nu_{21})} & 0 \\
0 & 0 & 0.5 \cdot G \cdot \frac{i_{12} + i_{21}}{2}
\end{bmatrix}
$$

We recognize these formula for the elasticity matrix $[Q]$:  

$$Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}}; \quad Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}}; \quad Q_{12} = Q_{21} = v_{21} \cdot Q_{11} = v_{12} \cdot Q_{22}; \quad Q_{66} = G \quad (11)$$

3.2.1. Manual calculation

- $d_{11} = \frac{10\,700\,N/mm^2 \cdot 20mm}{1 - 0.02 \cdot 0.51} = 216.2 \cdot 10^3 kN/m$
- $d_{22} = \frac{430N/mm^2 \cdot 20mm}{1 - 0.02 \cdot 0.51} = 8.7 \cdot 10^3 kN/m$
- $d_{12} = d_{21} = 0.02 \cdot 216.2 \cdot 10^3 kN/m = 4.3 \cdot 10^3 kN/m$
- $d_{66} = 620N/mm^2 \cdot 20mm = 12.4 \cdot 10^3 kN/m$

$$D_{11} = \frac{10\,700\,N/mm^2 \cdot (20mm)^3}{12(1 - 0.02 \cdot 0.51)} = 7.21kNm$$

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\[ D_{22} = \frac{370N/mm^2 \cdot (20mm)^3}{12(1 - 0.02 \cdot 0.51)} = 0.29kNm \]

\[ D_{12} = D_{21} = 0.057 \cdot 7.21kNm = 0.14kNm \]

\[ D_{66,av} = 0.5 \cdot 620N/mm^2 \cdot 0.5 \cdot \left( \frac{(20mm)^3}{6} + \frac{(20mm)^3}{6} \right) = 87.3kNm \]

### 3.2.2. In Diamonds

These values should be entered manually in Diamonds.

**Notes**

- Since the real shape of the plate is not known by Diamonds, no stresses nor reinforcement can be calculated.
- If the principal axes of orthotropic are rotated over an angle \( \theta \) in relation to the local \( x'z' \)-coordinate system, the elasticity matrix \([Q]\) will undergo a transformation to \([Q]_{\theta}\). In the transformed elasticity matrix \([Q]_{\theta}\) the components \(Q_{16,\theta}\) and \(Q_{26,\theta}\) are no longer equal to zero [6]. Diamonds assumes \(Q_{16} = Q_{26} = 0\), thus a plate with rotated principal axes of orthotropic is not possible with Diamonds.

\[
[Q] = \begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\]

\[
[Q]_{\theta} = \begin{bmatrix}
Q_{11,\theta} & Q_{12,\theta} & Q_{16,\theta} \\
Q_{12,\theta} & Q_{22,\theta} & Q_{26,\theta} \\
Q_{16,\theta} & Q_{26,\theta} & Q_{66,\theta}
\end{bmatrix}
\]
3.3. Orthotropic geometry (laminate) – orthotropic material

For a multi-ply laminate the relation for the sectional forces is a bit more complex than in §2.2. The purpose is to take the sum of the contributions of all laminate plies \( k \) [1] (Kubiak, 2013).

\[
\begin{bmatrix}
N_{xx} \\
N_{zz} \\
N_{xz}
\end{bmatrix} = \int_{t_k}^{t_{k-1}} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{zz} \\
\tau_{xz}
\end{bmatrix} dy = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{zz} \\
\tau_{xz}
\end{bmatrix} dy
\]

\text{sum of the contribution of the various layers}

\[
\begin{bmatrix}
M_{xx} \\
M_{zz} \\
M_{xz}
\end{bmatrix} = \int_{t_k}^{t_{k-1}} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{zz} \\
\tau_{xz}
\end{bmatrix} y dy = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{zz} \\
\tau_{xz}
\end{bmatrix} y dy
\]

\text{sum of the contribution of the various layers}

Resulting in these matrix equations:

\[
\begin{bmatrix}
N_{xx} \\
N_{zz} \\
N_{xz}
\end{bmatrix} = \begin{bmatrix}
d_{11} & d_{12} & d_{16} \\
d_{21} & d_{22} & d_{26} \\
d_{61} & d_{62} & d_{66}
\end{bmatrix} \begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{zz} \\
\gamma_{xz}
\end{bmatrix} + \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{21} & B_{22} & B_{26} \\
B_{61} & B_{62} & B_{66}
\end{bmatrix} \begin{bmatrix}
\kappa_{xx} \\
\kappa_{zz} \\
\rho_{xz}
\end{bmatrix}
\]

\text{(12)}

\[
\begin{bmatrix}
M_{xx} \\
M_{zz} \\
M_{xz}
\end{bmatrix} = \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{21} & B_{22} & B_{26} \\
B_{61} & B_{62} & B_{66}
\end{bmatrix} \begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{zz} \\
\gamma_{xz}
\end{bmatrix} + \begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{21} & D_{22} & D_{26} \\
D_{61} & D_{62} & D_{66}
\end{bmatrix} \begin{bmatrix}
\kappa_{xx} \\
\kappa_{zz} \\
\rho_{xz}
\end{bmatrix}
\]

\text{(13)}

The components in the matrix \([d], [B] \) and \([D] \) are calculated with the formula:

\[
d_{ij} = \sum_{k=1}^{n} (Q_{ij})_k (y_k - y_{k-1})
\]

\text{(14)}

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (Q_{ij})_k (y_k^2 - y_{k-1}^2)
\]

\text{(15)}

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{n} (Q_{ij})_k (y_k^3 - y_{k-1}^3)
\]

\text{(16)}

The 3x3 matrix \([d] \) and \([D] \) in equations (12) and (13) are again the membrane and bending stiffness matrix. In literature the matrix \([d] \) is referred to as matrix \([A] \). The 3x3 matrix \([B] \) in equations (12) and...
(13) is called the coupling matrix and relates the in-plane forces and the out-of-plane deformations. \( Q_{ij} \) in equations (14), (15) and (16) is the relevant component of the elasticity matrix. \( y \) is the distance from the outer edge of a ply towards the mid-plane.

### 3.3.1. Manual calculation

The elasticity matrix for a 0° ply is (11):

\[
Q_{11} = \frac{E_1}{1 - v_{12} v_{21}} = \frac{10700 \text{N/mm}^2}{1 - 0.51 \cdot 0.02} = 10810 \text{N/mm}^2
\]

\[
Q_{22} = \frac{E_2}{1 - v_{12} v_{21}} = \frac{430 \text{N/mm}^2}{1 - 0.51 \cdot 0.02} = 434 \text{N/mm}^2
\]

\[
Q_{12} = v_{21} \cdot Q_{11} = 0.02 \cdot 12163 \text{N/mm}^2 = 222 \text{N/mm}^2
\]

\[
Q_{66} = G = 620 \text{N/mm}^2
\]

\[
Q_{16} = Q_{26} = 0
\]

\[
[Q]_{0°} = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}
\]

\[
[Q]_{0°} = \begin{bmatrix}
10810 & 222 & 0 \\
222 & 434 & 0 \\
0 & 0 & 620
\end{bmatrix}
\]

In the elasticity matrix for a 90° ply, \( Q_{11} \) and \( Q_{22} \) switched places since the mean direction is rotated 90°:

\[
[Q]_{90°} = \begin{bmatrix}
Q_{22} & Q_{12} & Q_{16} \\
Q_{12} & Q_{11} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}
\]

\[
[Q]_{90°} = \begin{bmatrix}
434 & 222 & 0 \\
222 & 10810 & 0 \\
0 & 0 & 620
\end{bmatrix}
\]

The first component \( d_{11} \) of the membrane stiffness is calculated with equation (14):

\[
d_{11} = Q_{11} (30 \text{mm} - 10 \text{mm}) + Q_{22} (10 \text{mm} - (-10 \text{mm})) + Q_{12} (10 \text{mm} - (-10 \text{mm})) + 10810 (10 \text{mm} - (-30 \text{mm}))
\]

\[
d_{11} = 441.1 \cdot 10^3 \text{kN/m}
\]

The other components can be found the same way:

\[
d_{22} = 233.6 \cdot 10^3 \text{kN/m}
\]

\[
d_{12} = 13.3 \cdot 10^3 \text{kN/m}
\]

\[
d_{66} = 37.2 \cdot 10^3 \text{kN/m}
\]

The first component \( D_{11} \) of the membrane stiffness is calculated with equation (16):

\[
D_{11} = \frac{1}{3} \left[ Q_{11} (30^3 - 10^3) + Q_{22} (10^3 - (-10)^3) + Q_{12} ((-10)^3 - (-30)^3) \right]
\]

\[
D_{11} = \frac{1}{3} \left[ 10810 (30^3 - 10^3) + 434 (10^3 - (-10)^3) + 10810 ((-10)^3 - (-30)^3) \right] = 187.7 \text{kNm}
\]

The other components can be found the same way:
\[ D_{22} = 14.7 \text{kNm} \]
\[ D_{12} = 4.0 \text{kNm} \]
\[ D_{66} = 11.2 \text{kNm} \]

3.3.2. In Diamonds

These values should be entered manually in Diamonds.

**Notes:**

- Since the real shape of the plate is not known by Diamonds, no stresses nor reinforcement can be calculated.
- Pay attention when you calculate the stress: The strain varies linearly across the tickets. However, the stiffness properties (and thus the stresses) are discontinuous from one layer to the next!

![Figure 5: Variation of strain and stress in a hypothetical 3-ply laminate](image)

- The stiffness matrix in Diamonds assumes [1]:
  - \( d_{16} = d_{26} = 0 \)
  - \( D_{16} = D_{26} = 0 \)
  - \( [B] = 0 \)

This holds true, if all plies are orientated at 0° or 90° and when the composition of the plies is symmetrical.
• The calculated values can be verified with the tool from the eFunda website: http://www.efunda.com/formulae/solid_mechanics/composites/calc_ufrp_abd_go.cfm
• For the stiffness matrix of CLT-plates from Stora Enso, please check this website. https://forums.autodesk.com/autodesk/attachments/autodesk/2053/1646/1/15.02.20_StiffnessMatrix.pdf
4. References


https://www.ecourses.ou.edu/cgi-bin/eBook.cgi?doc=&topic=me&chap_sec=01.4&page=theory

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